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Effective Potential for the Reaction-Diffusion-Decay System

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In previous work we have developed a general method for casting stochastic partial differential equations (SPDEs) into a functional integral formalism, and have derived the one-loop effective potential for these systems. In this paper we apply the same formalism to a specific field theory of considerable interest, the reaction-diffusion-decay system. When this field theory is subject to white noise we can calculate the one-loop effective potential (for arbitrary polynomial reaction kinetics) and show that it is one-loop ultraviolet renormalizable in 1, 2, and 3 space dimensions. For specific choices of interaction terms the one-loop renormalizability can be extended to higher dimensions. We also show how to include the effects of fluctuations in the study of pattern formation away from equilibrium, and conclude that noise affects the stability of the system in a way which is calculable.

KEY WORDS: Effective potential; reaction, diffusion, decay.

1. INTRODUCTION

Geometrical patterns are ubiquitous: From galaxies to living systems, examples abound where a particular spatial distribution of some material is preferred versus others out of a seemingly unlimited variety. In many

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cases, these patterns are successfully described by systems of coupled parabolic non-linear partial differential equations. This is the case, for example, in chemical kinetics, where such equations summarize the space-time evolution of chemical species diffusing and reacting in some confined geometrical region, which makes its presence felt in the boundary conditions for the problem. In this way chemical kinetics helps one to understand leopard spots, zebra bands or the radial structure of *Acetabularia*. (See Murray,[1] Walgraef,[2] and Ball[3] for discussions of many specific examples. Useful background references include van Kampen[4] and Gardiner.[5]) In these phenomena the values of "reaction constants" play a role which reminds one of the role played by coupling constants in determining the vacuum (or ground) state in a quantum field theory undergoing spontaneous symmetry breaking, or in the description of phase transitions in condensed matter systems. This is not surprising, since in the presence of spatial non-equilibrium patterns[6] "it is often the case that the continuous symmetry of the system becomes spontaneously broken." Because of the above, two questions come immediately to mind: (a) what is the effect on an existing pattern of the elimination of fast degrees of freedom? and, (b) how do fluctuations affect the stability of an established pattern? These two questions are, of course, formally related since both phenomena manifest themselves through noise added to the otherwise deterministic equations describing the formation of these patterns. Providing answers to these questions opens the door to the study of complex phenomena where either there is no precise explicit knowledge of many of the microscopic details, or else where unexpected external perturbations and disturbances show up, and one is nevertheless interested in an explicit understanding of the system at long wavelengths. An example is in population ecology, where the presence of illegal hunting, or the accidental introduction of some (apparently minor) contaminant, can produce major ecological shifts.

The most appropriate tool to study patterns of symmetry is, of course, the notion of a potential. If some notion of potential is available, then analysis of its extrema leads to the identification of the stable and metastable vacua for the system. From there, given the values of the couplings in the system, one determines the ground state for the system. This analysis is performed in equilibrium, but the problem in pattern formation, or in reaction-diffusion systems, is that these systems are away from equilibrium, often far away from equilibrium, and the usual notion of "potential," "vacuum state," and allied concepts are no longer available. Recently, however, we have introduced[7] a notion of "effective potential" which generalizes the standard (Quantum Field Theory) notion of an effective potential to systems away from equilibrium. The generalization (as discussed below) is such that a very clean and clear parallel can be established

with the situation in Quantum Field Theory, and a potential is constructed which has two major and distinct pieces: a "classical" contribution and a "fluctuation" contribution. The "classical" contribution plus the "fluctuation" contribution determine the "vacuum state" of the system, and therefore this effective potential allows the calculation of the effects of the fluctuations on the ground state of the system. The "minimalist formalism" we developed in ref. 7 is to be contrasted with the traditional Martin–Siggia–Rose (MSR) formalism with its additional conjugate fields.[8, 9] One difference between the MSR approach and the "minimalist formalism" can be immediately appreciated at a calculational level, in terms of the structure of the diagramatic Feynman rules (see Appendix C). Each approach has its own advantages and disadvantes depending on the particular application one has in mind. For the purposes of calculating the effective potential, we have found the minimal formalism best suited for the task. The minimalist formalism is an extension and outgrowth of the Onsager–Machlup approach,[10, 11] and exhibits similarities to the analysis of Crisanti and Marconi.[12]

The fluctuation dependent piece of the potential involves integrations over the frequency and momentum domains. These integrals require the introduction of a cutoff, which, through a Wilsonian-style procedure, leads to a scale-dependence of the parameters of the reaction-diffusion-decay system; and therefore has an effect on the type of instability (and associated patterns) which controls the behavior of the system. The full "effective potential" then constitutes a superb tool to incorporate the effects of nonlinearities and fluctuations in the patterns produced by systems away from equilibrium of the reaction-diffusion type.

In this paper, after a brief discussion of the "effective potential" in general, and a description of how the noise amplitude plays the role of a loop counting parameter, we specialize to the case of real reaction-diffusion-decay systems subject to real additive noise. The "potential" is calculated for $d = 0, 1, 2$, and 3 spatial dimensions, and the special case of monomial interactions is given separate treatment. We also provide specialized discussions for higher dimensions and general colored noises. Then, as an application of the previous results, we study the effects of Gaussian white noise in both Hopf and Turing bifurcations by computing appropriate quantities without fluctuations and with fluctuations. We find that the effect of noise is to shift the symmetric states of the system, *as well* as to change the nature of the linear instabilities that may occur as perturbations around these new states. We end by offering some conclusions.

To fix the notation: In a companion paper,[7] we discussed classical field theories subject to additive stochastic noise $\eta(\vec{x}, t)$ described by the equation

$$D\phi(\vec{x}, t) = F[\phi(\vec{x}, t)] + \eta(\vec{x}, t) \tag{1}$$

Here D is any linear differential operator, involving arbitrary time and space derivatives, that does *not* explicitly involve the field ϕ. The function $F[\phi]$ is any forcing term, generally nonlinear in the field ϕ. These stochastic partial non-linear differential equations (SPDEs) can be studied using a functional integral formalism which makes manifest the deep connections with quantum field theories (QFTs). Methods of quantum field theory have been sucessfully applied before to classical statistical systems out of equilibrium, but in other *contexts* and with other objectives in mind. A method has been developed that takes a classical master equation to a continuum field theory.[13–15] This second-quantized formalism requires detailed knowledge of the microcoscopic master equation, and its "end-product" is the stochastic PDE for the coarse-grained degrees-of-freedom and the noise correlation functions.[16–21] In our approach, on the other hand, we *start* off with the stochastic PDE plus the "a priori given" noise correlation functions and apply field theory methods to map this *phenomenological* equation to a generating functional which allows us to define the effective action and the effective potential, which is the main objective of the present paper.

In ref. 7 we showed that if the noise is translation-invariant and Gaussian, it is possible to split its two-point function into an *amplitude* $\mathscr{A}$ and a *shape* function $g_2(x, y)$, as follows

$$G_\eta(x, y) \stackrel{\text{def}}{=} \mathscr{A} g_2(x-y) \tag{2}$$

with the *convention* that

$$\int \mathrm{d}^d\vec{x}\, \mathrm{d}t\, g_2^{-1}(\vec{x}, t) = 1 = \tilde{g}_2^{-1}(\vec{k}=\vec{0}, \omega=0) \tag{3}$$

Then the one-loop effective potential for the SPDE is[7]

$$\begin{aligned}\mathscr{V}[\phi; \phi_0] = &\frac{1}{2} F^2[\phi] + \frac{1}{2}\mathscr{A} \int \frac{\mathrm{d}^d\vec{k}\, \mathrm{d}\omega}{(2\pi)^{d+1}} \\ &\times \ln\left[1 + \frac{\tilde{g}_2(\vec{k}, \omega)\, F[\phi](\delta^2 F/\delta\phi\, \delta\phi)}{(D^\dagger(\vec{k}, \omega) - (\delta F/\delta\phi)^\dagger)(D(\vec{k}, \omega) - (\delta F/\delta\phi))}\right] \\ &- (\phi \to \phi_0) + O(\mathscr{A}^2)\end{aligned} \tag{4}$$

Here ϕ_0 is any convenient background field. The above result is qualitatively similar to the one-loop effective potential for scalar QFT:[22–25]

$$\mathscr{V}[\phi; \phi_0] = V(\phi) + \frac{1}{2}\hbar \int \frac{\mathrm{d}^d\vec{k}\, \mathrm{d}\omega}{(2\pi)^{d+1}} \ln\left[1 + \frac{\delta^2 V/(\delta\phi\, \delta\phi)}{\omega^2 + \vec{k}^2 + m^2}\right] - (\phi \to \phi_0) + O(\hbar^2) \tag{5}$$

as can be seen by simply comparing Eqs. (4) and (5). Moreover, as argued in ref. 7, this effective potential for SPDEs inherits many of the interesting features of the effective potential for QFTs. In particular, minima of the effective potential for a given SPDE correspond to homogeneous and static solutions of the stochastic equations of motion (that is, homogeneous and static expectation values of the stochastically driven field ϕ, with the averaging done with respect to noise realizations).[7]

In a second paper,[26] we applied this formalism to the Kardar–Parisi–Zhang (KPZ) equation (relevant in the study of surface growth phenomena and cosmological large-scale structure formation) obtaining an interesting ground state structure (including dynamical symmetry breaking). In the current paper we perform a similar analysis for the reaction-diffusion-decay system described by the equation:

$$\left(\frac{\partial}{\partial t} - \nu\vec{\nabla}^2\right)\phi = P_0 - \gamma\phi + P(\phi) + \eta \tag{6}$$

This equation can be used, for instance, as a model to describe the dynamics and spatial distribution of the concentration of a chemical reagent, when it is subject to both diffusion (via ν) and decay (via γ). $P(\phi)$ is some ultra-local function of the concentration (typically a polynomial in the concentration, but not always) and it represents the reaction kinetics.[1–5] However, it appears that this equation might not be phenomenologically suitable for modelling pair-reaction kinetics (i.e., $\phi + \phi \to 0$), since the stochastic PDE that is *derived* from the microscopic master equation is *complex* and contains *imaginary* noise.[16, 17] On the other hand, for standard Gribov processes (particle clustering reactions) one derives a real stochastic PDE with real additive noise. We therefore see that the class considered in (6) still covers a wide range of interesting phenomena.

We have included a "tadpole term" P_0 since, as we will see, its presence is essential for the consistency of the ultraviolet renormalization program. Many examples of reaction-diffusion equations abound in the literature. By way of example, we mention just two model equations that the reader may wish to keep in mind. One goes under the name of "amplitude equation," generally complex, resulting from reducing a (non-stochastic) reaction-diffusion equation in the vicinity of an instability point:

$$\partial_t A = \nu\vec{\nabla}^2 A + f(|A|^2)\, A \tag{7}$$

where the diffusion coefficient can be complex: $\nu = \nu_R + i\nu_I$. The study of amplitude equations is very useful for determining the basic geometry of patterns that can emerge near the instability point.[2, 3] This equation

clearly involves only one field degree of freedom, but the generalization is straightforward. For several species of, e.g., chemical reactant the field (concentration) ϕ is promoted to a vector $\phi_i(\vec{x}, t)$. The diffusion coefficient and decay rate are then promoted to matrices, the noise to a vector, and the reaction kinetic function $P(\phi)$ to a vector-valued functional with tensorial coefficients.

$$\left(\delta_i{}^j \frac{\partial}{\partial t} - v_i{}^j \vec{\nabla}^2\right) \phi_j = (P_0)_i - \gamma_i{}^j \phi_j + P_i(\phi_j) + \eta_i \tag{8}$$

Typical examples of this sort involving two components go under the heading of "activator-inhibitor" models of e.g., biological pattern formation, a noise-free example of which is provided by the Gierer–Meinhardt mechanism:

$$\partial_t A = D_A \vec{\nabla}^2 A + k_1 - k_2 A + \frac{k_3 A^2}{B} \tag{9}$$

$$\partial_t B = D_B \vec{\nabla}^2 A + k_4 A^2 - k_5 B \tag{10}$$

Here $\phi_1 = A$ is the activator and $\phi_2 = B$ is the inhibitor. The k_i are reaction constants.

Finally, we note that these systems provide a viable framework for the notion of "self-organizing systems."

The remainder of this paper is organized as follows. In Section 2, following the general procedure developed in ref. 7, we construct the complete one-loop effective potential for the class of reaction-diffusion equations in (6). Once this is done, we carry out a detailed analysis of the resulting potential as a function of spatial dimension for $d = 0, 1, 2, 3$. We keep the analysis as general as possible and calculate the effective potential for *arbitrary* reaction polynomials and in *arbitrary* spatial dimensions. Specialization to concrete reaction kinetics is immediate by substituting in a specific form for the polynomial, and choosing the space dimension. Special attention is also paid to monomial interactions, the case of higher spatial dimensions, and the case of correlated Gaussian noise. In Section 3 we turn to a discussion of the impact that noise and fluctuations can have on the onset of instabilities and pattern formation. We conclude in Section 4 with a discussion of our results. Certain technical issues having to do with functional Jacobian determinants are collected in Appendixes A and B. The general Feynman rules for the reaction-diffusion equation (6) are presented in Appendix C. An integral needed in the computation of the one-loop effective potential is calculated in Appendix D. Finally, it should be borne in mind that the main body of the paper adopts the so-called *Stratonovich*

calculus–Appendix E indicates the changes that are required if one wishes to consider the so-called *Ito calculus*.

2. EFFECTIVE POTENTIAL: REACTION-DIFFUSION-DECAY SYSTEMS

We begin this section by computing the one-loop effective potential associated to the class of reaction-diffusion equations (6). We first carry out the computations for arbitrary Gaussian noise and arbitrary spatial dimension, only later specializing to white Gaussian noise and examining the particular features of the effective potential for dimensions $d = 0, 1, 2, 3$.

To avoid unnecessary clutter, let us re-write Eq. (6) as

$$\left(\frac{\partial}{\partial t} - \nu \vec{\nabla}^2\right) \phi = P(\phi) + \eta \tag{11}$$

Here $P(\phi)$ is an arbitrary polynomial in the field ϕ. Any tadpole contribution P_0, as well as any decay term $-\gamma\phi$, have now for convenience been subsumed into $P(\phi)$.

To apply the general analysis provided in ref. 7, which led to Eqs. (1)–(4) above, to a homogeneous and static background field, $\phi(\vec{x}, t) = \text{constant}$, we simply make the identifications

$$F[\phi] \to P(\phi); \qquad \frac{\delta F}{\delta\phi(x)} \to P'(\phi); \qquad \frac{\delta^2 F}{\delta\phi(x)\,\delta\phi(y)} \to P''(\phi) \tag{12}$$

as follows by comparing Eqs. (1) and (11). At tree-level (zero-loop) in the loop-counting parameter $\mathscr{A}$, the equations of motion become[7]

$$\frac{\delta F^\dagger}{\delta\phi} F[\phi] = J \;\to\; P'(\phi)\, P(\phi) = J \tag{13}$$

This is a polynomial equation for ϕ and therefore has a finite number of roots. In particular, for $J = 0$ let ϕ_0 be *one* of the roots of $P'(\phi)\, P(\phi) = 0$. This polynomial always has at least one real root. [Proof: let $P(\phi)$ be of degree n, then $P^2(\phi)$ is of degree $2n$, and $P'(\phi)\, P(\phi)$ is of degree $2n-1$, which is always odd. Thus $P'(\phi)\, P(\phi)$ must cross the abscissa at least once, so there must be at least one real zero.]

The zero-loop effective potential is now

$$\mathscr{V}_{\text{zero-loop}}[\phi; \phi_0] = \tfrac{1}{2}[P^2(\phi) - P^2(\phi_0)] \tag{14}$$

This zero-loop effective potential is both a generalization (because it includes the effects of non-linearities) and a specialization (because it treats static fields) of the Onsager–Machlup action for stochastic mechanics.[7, 10, 11] We start the one-loop computation by noting that for the reaction-diffusion-decay system the linear differential operator needed is given by

$$D - \frac{\delta F}{\delta \phi} = \partial_t - v\,\vec{\nabla}^2 - P'(\phi) \;\rightarrow\; -i\omega + v\vec{k}^2 - P'(\phi) \tag{15}$$

in configuration and Fourier variables. We then have for the adjoint quantity

$$D^\dagger - \frac{\delta F^\dagger}{\delta \phi} = -\partial_t - v\,\vec{\nabla}^2 - P'(\phi) \;\rightarrow\; +i\omega + v\vec{k}^2 - P'(\phi) \tag{16}$$

so that

$$\left(D^\dagger - \frac{\delta F^\dagger}{\delta \phi}\right)\left(D - \frac{\delta F}{\delta \phi}\right) = -\partial_t^2 + [\,v\,\vec{\nabla}^2 + P'(\phi)\,]^2 \rightarrow +\omega^2 + [\,v\vec{k}^2 - P'(\phi)\,]^2 \tag{17}$$

Using the previous equations, which are specializations of the general ones presented in ref. 7, the one-loop effective potential is given by

$$\mathscr{V}[\,\phi;\phi_0\,] = \frac{1}{2}P^2(\phi) + \frac{1}{2}\mathscr{A}\int \frac{\mathrm{d}^d\vec{k}\,\mathrm{d}\omega}{(2\pi)^{d+1}} \ln\left[\,1 + \frac{\tilde{g}_2(\vec{k},\omega)\,P(\phi)\,P''(\phi)}{\omega^2 + [\,v\vec{k}^2 - P'(\phi)\,]^2}\right] - (\phi \rightarrow \phi_0) + O(\mathscr{A}^2) \tag{18}$$

Equivalently

$$\mathscr{V}[\,\phi;\phi_0\,] = \frac{1}{2}P^2(\phi) + \frac{1}{2}\mathscr{A}\int \frac{\mathrm{d}^d\vec{k}\,\mathrm{d}\omega}{(2\pi)^{d+1}} \times \ln\left[\frac{\omega^2 + [\,v\vec{k}^2 - P'(\phi)\,]^2 + \tilde{g}_2(\vec{k},\omega)\,P(\phi)\,P''(\phi)}{\omega^2 + [\,v\vec{k}^2 - P'(\phi)\,]^2}\right] - (\phi \rightarrow \phi_0) + O(\mathscr{A}^2) \tag{19}$$

At this stage we should be explicit about some technical details: First, the way we have chosen to treat the functional Jacobian is equivalent to choosing the *Stratonovich calculus* for the stochastic noise.[7] The modifications attendant on the choice of the *Ito calculus* are in some ways a simplification of the current procedure, but in other ways lead to additional

technical complications. Appendix E discusses some features of the Ito calculus. Second, in deriving the formulae above it has been assumed (in performing the functional integrations) that the field ϕ is unrestricted, and can take on all values from $-\infty$ to $+\infty$. Strictly speaking, this is of course not the case if ϕ represents a concentration, but this is not a serious restriction. One can deal with this either by (1) choosing the forcing term $P(\phi)$ to strongly suppress negative values of ϕ, and then taking a suitable limit, or more prosaicly (2) by realising that in making the one-loop approximation we have already assumed that fluctuations are in some sense small, so that if we look at quadratic fluctuations around some positive value of the background field ϕ_0, then the error made in letting $\delta\phi = \phi - \phi_0$ range over the entire real line is a higher order effect [at least $O(\mathscr{A}^2)$].

This is as far as we can go *without making any further assumptions about the additive noise*. For instance, one standard choice is *temporally white*, which means delta function correlated in time, so that $\tilde{g}_2(\vec{k}, \omega) \to \tilde{g}_2(\vec{k})$ is a function of $\vec{k}$ only. Let us define $X^2 = [\,v\vec{k}^2 - P'(\phi)\,]^2 + \tilde{g}_2(\vec{k})\, P(\phi)\, P''(\phi)$ and $Y^2 = [\,v\vec{k}^2 - P'(\phi)\,]^2$. It is easy to see from its definition that Y^2 is real and positive, and so is Y. If X^2 is positive, we can make use of the standard integral identity [X and Y are positive, see ref. 27, Eq. (4.222.1)], namely

$$\int_{-\infty}^{+\infty} \mathrm{d}\omega \ln\left(\frac{\omega^2 + X^2}{\omega^2 + Y^2}\right) = 2\pi(X - Y) \tag{20}$$

to re-write Eq. (19) as

$$\mathscr{V}[\,\phi;\phi_0\,] = \frac{1}{2} P^2(\phi) + \frac{1}{2}\mathscr{A} \int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} \left[\sqrt{[\,v\vec{k}^2 - P'(\phi)\,]^2 + \tilde{g}_2(\vec{k})\, P(\phi)\, P''(\phi)} - |\,v\vec{k}^2 - P'(\phi)\,|\,\right] - (\phi \to \phi_0) + O(\mathscr{A}^2)$$

If X^2 is real and negative, we can define $X^2 = -Z^2$, with Z a real positive number. In this case we must make use of the following integral (see Appendix D)

$$\int_{-\infty}^{+\infty} \mathrm{d}\omega \ln\left(\frac{\omega^2 - Z^2 \pm i\varepsilon}{\omega^2 + Y^2}\right) = 2\pi(\pm iZ - Y) \tag{21}$$

to get the appropriate result; here ε is a real, small, and positive number.

Averaging over the $\pm$ justifies the following prescription

$$\int_{-\infty}^{+\infty} \mathrm{d}\omega \ln\left(\frac{\omega^2 - Z^2}{\omega^2 + Y^2}\right) \to \int_{-\infty}^{+\infty} \mathrm{d}\omega\, \mathrm{Re}\left[\ln\left(\frac{\omega^2 - Z^2}{\omega^2 + Y^2}\right)\right] = -2\pi Y \tag{22}$$

This prescription is guaranteed to preserve the reality of the effective potential. In general we should write

$$\mathscr{V}[\phi;\phi_0] = \frac{1}{2}P^2(\phi) + \frac{1}{2}\mathscr{A}\int\frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} \times\{\mathrm{Re}[\sqrt{[v\vec{k}^2 - P'(\phi)]^2 + \tilde{g}_2(\vec{k})\,P(\phi)\,P''(\phi)}] - |v\vec{k}^2 - P'(\phi)|\} - (\phi\to\phi_0) + O(\mathscr{A}^2) \tag{23}$$

where we have used the ω-integral to recast (19) into (23). Remembering that $\lim_{\vec{k}\to\vec{0}}\tilde{g}_2(\vec{k}) = 1$ [recall (3)], it is clear that there are no infrared divergences ($\vec{k}\to\vec{0}$), at least for this effective potential at one-loop order. To investigate the ultraviolet (i.e., short-distance) behaviour, it is useful to re-express this in the form

$$\mathscr{V}[\phi;\phi_0] = \frac{1}{2}P^2(\phi) + \frac{1}{2}\mathscr{A}\int\frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}\,|v\vec{k}^2 - P'(\phi)| \times\left\{\mathrm{Re}\sqrt{1 + \frac{\tilde{g}_2(\vec{k})\,P(\phi)\,P''(\phi)}{[v\vec{k}^2 - P'(\phi)]^2}} - 1\right\} - (\phi\to\phi_0) + O(\mathscr{A}^2) \tag{24}$$

It is clear now that this effective potential will be finite provided the spatial part of the noise spectrum satisfies

$$\int\mathrm{d}^d\vec{k}\,\frac{\tilde{g}_2(\vec{k})}{|\vec{k}^2 + a|} < \infty \tag{25}$$

Thus, while the noise acts as the source of the fluctuations, it can also serve as the regulator to keep physical quantities UV-finite, as should be clear from (25). Indeed, whatever modifications one might make to the noise in the ultraviolet region will have no consequence for the long-wavelength or hydrodynamic limit. The noise is intended to model fluctuations above a certain limiting resolution length/time scale, thereby making its short distance behavior immaterial, in so far as one is interested in studying the long distance, long time asymptotic behavior of the stochastic model.[6]

[6] Since this point may cause some confusion to the reader, we briefly pause to belabour it: UV renormalizability is *not* the same as UV *finiteness* and we do not claim that the theory has to make sense at arbitrarily short distances. (If nothing else, in real chemical kinetics the interatomic spacing will provide a natural UV cutoff.) What UV renormalizability does is to sharply *limit* the number of relevant operators, so that the low energy theory (long distances, large times) is guaranteed to be relatively simple. Lack of UV renormalizability is not fatal for a theory, but does make life considerably more complicated. This phenomenon is known in the quantum field theory literature as "decoupling;" see e.g., Weinberg II.[22]

For definiteness, let us now take the spatial noise spectrum to be cutoff white, i.e.,

$$\tilde{g}_2(\vec{k}) = \tilde{g}_2(|\vec{k}|) = \Theta(\Lambda - k) \tag{26}$$

With this choice of noise, we can Taylor expand the square root of Eq. (24) in the ultraviolet regime to see that there is a divergent term proportional to $P(\phi)\, P''(\phi)\, \Lambda^{d-2}$, and a subdominant divergent term proportional to $P(\phi)\, P'(\phi)\, P''(\phi)\, \Lambda^{d-4}$. Since the classical (tree-level) potential is just $P^2(\phi)$, to have any hope of absorbing the infinities into the bare action we must have $d < 4$. That is: the reaction-diffusion-decay system, subject to white noise and for *any* polynomial $P(\phi)$, is one-loop ultraviolet renormalizable *only* in 0,1,2, and 3 space dimensions. In 0 space dimensions the reaction-diffusion-decay system reduces to a Langevin reaction-decay system which is still interesting, (see below). In $d = 1$ the reaction-diffusion-decay system is in fact one-loop finite. (Strictly speaking the claim of one-loop renormalizability also requires the investigation of the wavefunction renormalization; this is beyond the scope of the present paper. For related discussion see ref. 28.) The assertion that arbitrary polynomial reaction kinetics can be renormalizable in low dimensions should not (with hindsight) be alarming. After all, exactly the same thing happens for quantum field theories in $d = 2$ spacetime dimensions, where $P(\phi)_2$ is renormalizable for arbitrary polynomials. If we restrict the form of the polynomial occurring in the reaction-diffusion-decay system, we can have one-loop renormalizable theories in a higher dimensions. We will come back to this particular point later.

To be more explicit, we expand the unrenormalized one-loop effective potential as follows

$$\mathscr{V}[\phi; \phi_0] = \frac{1}{2} P^2(\phi) + \frac{1}{2} \mathscr{A} \int \frac{\mathrm{d}^d \vec{k}}{(2\pi)^d} \sum_{n=1}^{\infty} \left\{ \binom{1/2}{n} \frac{[P(\phi)\, P''(\phi)]^n}{[v\vec{k}^2 - P'(\phi)]^{2n-1}} \right\} - (\phi \to \phi_0) + O(\mathscr{A}^2) \tag{27}$$

This expansion only makes sense if $|P(\phi)\, P''(\phi)| < [v\vec{k}^2 - P'(\phi)]^2$ for every value of $|\vec{k}|$. This requires both $P'(\phi) < 0$ and $|P(\phi)\, P''(\phi)| < [P'(\phi)]^2$. The relevant integrals, after a rescaling, are[7]

$$\mathscr{I}_{RDD}(n, d) \stackrel{\text{def}}{=} \int_0^\infty \mathrm{d}x\, x^{d-1}(1 + x^2)^{1-2n} = \frac{1}{2} B(d/2, 2n - 1 - d/2) = \frac{\Gamma(d/2)\, \Gamma(2n - 1 - d/2)}{2\Gamma(2n - 1)} \tag{28}$$

[7] See ref. 27, Section (8.380.3).

These integrals converge for $n > (d+2)/4$. For $n \leqslant (d+2)/4$ we should introduce appropriate counterterms. Of course, the one-loop effective potential may make perfectly good sense even when this Taylor series expansion is problematic. By this we simply mean that the one-loop contribution can be finite for values of the parameters and momentum that lie formally outside the radius of convergence of the above series: one would then obviously not proceed by expanding and integrating term-wise, as we have done here.

Finally we should remind the reader that even if the generic reaction-diffusion-decay system is non-renormalizable for $d \geqslant 4$, this does not mean that such theories are completely useless. (Though it must be admitted that the number of physically relevant examples in four or more space dimensions is rather limited, [see for example variants on the idea of Kaluza-Klein theory], the non-renormalizability *per se* is not the issue.) All that non-renormalizability implies is that the theory must be viewed as an "effective field theory" that must include many more terms in the effective action than naively arise in the zero-loop approximation. These new terms carry with them additional (typically dimensionful) coupling constants, and these coupling constants govern the range of validity of the effective field theory. See ref. 22 for a modern discussion of effective field theories in the QFT context. In statistical mechanics language, the universality class of an effective field theory is much more complicated than would be naively deduced from the zero-loop approximation.

We next plunge into a discussion of the above as a function of the *spatial* dimension, d.

2.1. Reaction-(Diffusion)-Decay: $d = 0$

In $d = 0$ space dimensions, there is of course no diffusion, and the system reduces to a simple Langevin reaction-decay system. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi = P(\phi) + \eta \tag{29}$$

Specific examples of this behaviour include the noisy logistic equation (with *additive* noise)

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi = r\phi\left(1 - \frac{\phi}{\phi_*}\right) + \eta \tag{30}$$

and the noisy Lotka–Voltera equation (used as a model for predator-prey interactions)

$$\frac{\mathrm{d}}{\mathrm{d}t} N = N(a + bP) + \eta_N \tag{31}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} P = P(cN - d) + \eta_P \tag{32}$$

More generally, one can easily construct noisy versions of standard toy models as the Michaelis–Menten model for enzymatic autocatalytic reactions, the Goodwin switch (a model of feedback control), the Brusselator, the Fitzhugh–Naguno model of nerve potentials, or the Field–Noyes model for oscillating reactions.[1]

In $d = 0$ there are tremendous simplifications in the general formalism. For time translation-invariant Gaussian noise it follows that

$$\mathscr{V}[\phi; \phi_0; d = 0] = \frac{1}{2} P^2(\phi) + \frac{1}{2} \mathscr{A} \int \frac{\mathrm{d}\omega}{(2\pi)} \times \ln\left[1 + \frac{\tilde{g}_2(\omega)\, P(\phi)\, P''(\phi)}{\omega^2 + [P'(\phi)]^2} \right] - (\phi \to \phi_0) + O(\mathscr{A}^2) \tag{33}$$

Equivalently,

$$\mathscr{V}[\phi; \phi_0; d = 0] = \frac{1}{2} P^2(\phi) + \frac{1}{2} \mathscr{A} \int \frac{\mathrm{d}\omega}{(2\pi)} \times \ln\left[\frac{\omega^2 + [P'(\phi)]^2 + \tilde{g}_2(\omega)\, P(\phi)\, P''(\phi)}{\omega^2 + [P'(\phi)]^2} \right] - (\phi \to \phi_0) + O(\mathscr{A}^2)$$

Just as in the case of field theory ($d \geqslant 1$) this is as far as we can go without making any further assumptions about the noise. For instance, *temporally white* noise implies $\tilde{g}_2(\omega) \to 1$. Integrating over frequencies and using the integral (20) supplemented by (21) and (22), yields

$$\mathscr{V}[\phi; \phi_0; d = 0] = \tfrac{1}{2} P^2(\phi) + \tfrac{1}{2} \mathscr{A} \{ \mathrm{Re} \sqrt{[P'(\phi)]^2 + P(\phi)\, P''(\phi)} - \sqrt{[P'(\phi)]^2} \} - (\phi \to \phi_0) + O(\mathscr{A}^2)$$

which can be re-written as

$$\mathscr{V}[\phi;\phi_0;d=0]=\tfrac{1}{2}P^2(\phi)+\tfrac{1}{2}\mathscr{A}\{\mathrm{Re}\sqrt{\tfrac{1}{2}[P^2(\phi)]''}-|P'(\phi)|\}$$
$$-(\phi\to\phi_0)+O(\mathscr{A}^2) \tag{34}$$

Note that from a field theory point of view a SDE in $0+1$ dimensions is *almost* quantum mechanics, and one-loop physics is *almost* semi-classical JWKB physics. To see what we mean by this, consider the quantum mechanics of a system governed by the Lagrangian

$$L=\frac{1}{2}m\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2-V(x) \tag{35}$$

The techniques more usually applied to quantum *field theory* can also be applied to quantum *mechanics* to obtain a quantum mechanical effective potential

$$\mathscr{V}_{QM}[x;x_0]=V(x)+\frac{1}{2}h\,\mathrm{Re}\left[\sqrt{\frac{V''(x)}{m}}\right]-(x\to x_0)+O(h^2) \tag{36}$$

This effective potential has the standard interpretation of being the minimum expectation value of the Hamiltonian operator, when extremized over stationary states satisfying $\langle\hat{x}\rangle=\vec{x}$. In this case there is a second possible interpretation in terms of the zero-point energy associated with the natural oscillation frequency $\Omega=\sqrt{V''/m}$, and it is in this sense that one-loop quantum mechanics is equivalent to semi-classical quantum mechanics. The reaction-(diffusion)-decay system is formally very similar to one-loop quantum mechanics with the replacement $V(x)\to\frac{1}{2}P(\phi)^2$; the only difference arising from the manner in which the Jacobian is treated (the $|P'(\phi)|$ term). In quantum mechanics two classically degenerate minima often have their degeneracy *broken via semi-classical effects*: from the discussion above, analogous phenomenon is seen to occur in stochastic mechanics.

(Note, however, that this is an analogy, not an identity. The SDE always gives rise to Weiner functional integrals, analogous to Euclidean quantum mechanics, instead of the Feynman functional integrals of quantum mechanics, and the SDE never exhibits the interference phenomena and complex amplitudes that are so characteristic of quantum mechanics.)

2.2. Reaction-Diffusion-Decay: $d=1$

Spatial structures and patterns in one dimension, such as for example the prenatal tail markings in *Genetta genetta* (common genet) can be

successfully modeled by one-dimensional reaction-diffusion equations. The reader is referred to the book by Murray for more concrete examples.[1]

For $d=1$ the relevant integral, though finite, is not analytically tractable. We are interested in evaluating

$$\begin{aligned}\mathscr{V}[\phi;\phi_0;d=1] = &\frac{1}{2}P^2(\phi)+\frac{1}{2\pi}\mathscr{A}\int_0^\infty \mathrm{d}k\{\mathrm{Re}\sqrt{[\nu k^2-P'(\phi)]^2+P(\phi)\,P''(\phi)}\\ &-\sqrt{[\nu k^2-P'(\phi)]^2}\}\\ &-(\phi\to\phi_0)+O(\mathscr{A}^2)\end{aligned} \tag{37}$$

By expanding in a power series, rescaling, integrating, and summing we obtain the following result

$$\begin{aligned}\mathscr{V}[\phi_0;\phi_0;d=1] = &\frac{1}{2}P^2(\phi)+\frac{1}{2\pi}\mathscr{A}\frac{[-P'(\phi)]^{3/2}}{\nu^{1/2}}\\ &\times\sum_{n=1}^{\infty}\binom{1/2}{n}\frac{\Gamma(1/2)\,\Gamma(2n-3/2)}{2\Gamma(2n-1)}\\ &\times\left(\frac{P(\phi)\,P''(\phi)}{[P'(\phi)]^2}\right)^n-(\phi\to\phi_0)+O(\mathscr{A}^2)\end{aligned} \tag{38}$$

This particular expansion only makes sense for $P'(\phi)<0$, which is the region of field configuration space in which small perturbations die away in the absence of noise.[8] Furthermore, the radius of convergence of the resulting sum is equal to one, so that this expression is limited to the region $PP''<(P')^2$. These limitations are not fundamental, but are artifacts of the expansion and integration procedure (the individual limits of first expanding and then integrating term by term do not commute) which must be taken into consideration.

It is easy to see that the integral of Eq. (37) is real, convergent, and well behaved for $P'(\phi)>0$ and $P(\phi)\,P''(\phi)>[P'(\phi)]^2$. Indeed, there is an exact (if rather formal) representation of this integral in terms of incomplete elliptic integrals of the first and second kinds. The integral can also be evaluated in terms of a ${}_3F_2$ generalized hypergeometric function and/or an assortment of complete Elliptic integrals.[9] Though exact, these formulations are too cumbersome to be useful and we can better understand the

[8] Inserting $\phi(t)=\phi_0+\delta\phi(t)$ into the noiseless version of (6), and expanding to first order in the time-dependent perturbation $\delta\phi(t)$ yields: $\mathrm{d}[\delta\phi(t)]/\mathrm{d}t=P'(\phi_0)\,\delta\phi(t)+O[(\delta\phi)^2]$. So for $P'(\phi_0)<0$ perturbations are damped in the absence of noise.

[9] For further details see ref. 27, section (9.14), p. 1045, and/or section (8.1).

general situation by rescaling the original integral, using $k = \sqrt{|P'(\phi)|/v}\, x$, to give

$$\mathscr{V}[\phi;\phi_0;d=1] = \frac{1}{2}P^2(\phi) + \frac{1}{2\pi}\mathscr{A}\frac{|P'(\phi)|^{3/2}}{v^{1/2}}\int_0^\infty \mathrm{d}x \times\left\{\mathrm{Re}\sqrt{(x^2\pm 1)^2 + \frac{P(\phi)\,P''(\phi)}{[P'(\phi)]^2}} - |x^2\pm 1|\right\} -(\phi\to\phi_0) + O(\mathscr{A}^2) \tag{39}$$

where $\pm = -\mathrm{sign}[P'(\phi)]$. Thus without any detailed calculations we know that the form of the effective potential is

$$\mathscr{V}[\phi;\phi_0;d=1] = \frac{1}{2}P^2(\phi) + \frac{1}{2\pi}\mathscr{A}\frac{|P'(\phi)|^{3/2}}{v^{1/2}}F_\pm\left[\frac{P(\phi)\,P''(\phi)}{[P'(\phi)]^2}\right] -(\phi\to\phi_0) + O(\mathscr{A}^2) \tag{40}$$

with $F_\pm[z]$ the function

$$F_\pm[z] = \int_0^\infty \mathrm{d}x\{\mathrm{Re}\sqrt{(x^2\pm 1)^2 + z} - |x^2\pm 1|\} \tag{41}$$

such that $F_\pm[z=0]=0$. It is $F_+[z]$ that corresponds to the case $P'(\phi)<0$ discussed above [see Eq. (38)]. The case $F_-[z]$ is trickier as there is no simple Taylor series expansion around $z=0$ (at least not in integer powers of z). The overall shape of these functions can be seen in Figs. 1 and 2.

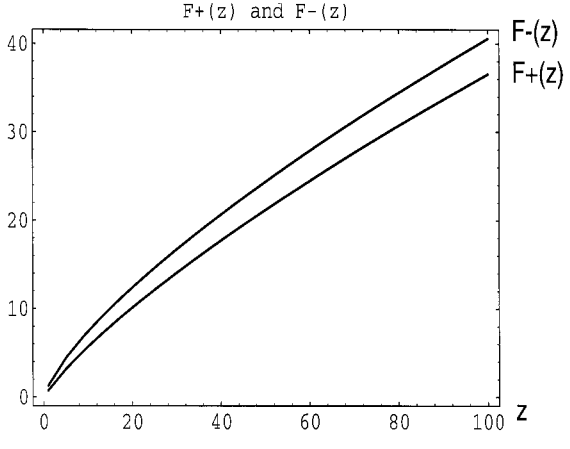

Fig. 1. Plot of $F_\pm(z)$ for positive values of z, from zero to 100.

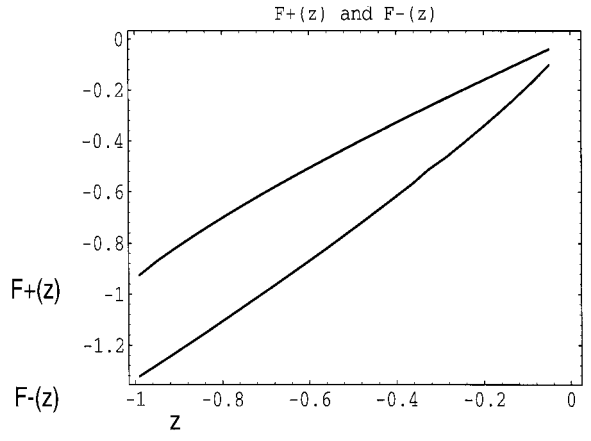

Fig. 2. Plot of $F_{\pm}(z)$ for negative values of z, from -1.00 to zero.

Near a *stable fixed point* one has $P'(\phi) < 0$, as mentioned above. We can use either Eq. (38) or expand Eq. (41) for $F_{+}(z)$ to deduce that

$$\mathscr{V}_{\text{stable}}[\phi; \phi_0; d=1] = \frac{1}{2} P^2(\phi) + \frac{1}{2\pi} \frac{\mathscr{A}}{v^{1/2}} \frac{\Gamma(1/2)^2}{4} \frac{P(\phi)\, P''(\phi)}{[-P'(\phi)]^{1/2}}$$

$$-(\phi \to \phi_0) + O\left(\mathscr{A} \frac{P^2 (P'')^2}{(P')^{5/2}}\right) + O(\mathscr{A}^2) \qquad (42)$$

This is the universal behaviour near a stable fixed point of the PDE.

On the other hand, at a point of *neutral stability* for the original noise-free PDE we have $P'(\phi) \to 0$, corresponding to $z \to \infty$. Fortunately, the *large* z behaviour can be analyzed analytically. Some tedious integral analysis leads to the expressions

$$F_{\pm}(z) = A_{+\infty} z^{3/4}[1 + O(1/z)] = \frac{1}{6} \frac{\Gamma(1/4)^2}{\sqrt{\pi}} z^{3/4}(1 + O(1/z))$$

$$\to 1.23605\ldots\, z^{3/4}(1 + O(1/z)) \qquad \text{as} \quad z \to +\infty \qquad (43)$$

$$F_{\pm}(-z) = -A_{-\infty} z^{3/4}[1 + O(1/z)] = -\frac{1}{6} \frac{\Gamma(1/4)^2}{\sqrt{2\pi}} z^{3/4}(1 + O(1/z))$$

$$\to -0.87401\ldots\, z^{3/4}(1 + O(1/z)) \qquad \text{as} \quad z \to +\infty \qquad (44)$$

Thus near a point of *neutral stability* we find universal behaviour described by

$$\mathscr{V}_{\text{neutral}}[\phi;\phi_0;d=1]=\frac{1}{2}P^2(\phi)+\frac{1}{2\pi}\mathscr{A}A_{\pm\infty}\frac{[P(\phi)\,P''(\phi)]^{3/4}}{v^{1/2}}$$
$$-(\phi\to\phi_0)+O\left(\mathscr{A}P'(\phi)^2\sqrt[4]{P(\phi)\,P''(\phi)}\right)+O(\mathscr{A}^2) \tag{45}$$

Finally, near an *unstable fixed point* $P'(\phi)>0$. Then we must use Eq. (41) for $F_-(z)$. For $z>0$ a (not particularly obvious) series of manipulations leads to

$$\frac{\mathrm{d}F_-(z)}{\mathrm{d}z}=\frac{1}{2}\int_0^\infty\frac{\mathrm{d}x}{\sqrt{(x^2-1)^2+z}}=\frac{1}{2}\,\mathrm{Re}\left[\frac{1}{\sqrt{1+i\sqrt{z}}}\,\mathbf{K}\left(\sqrt{\frac{1-i\sqrt{z}}{1+i\sqrt{z}}}\right)\right]$$

Here $\mathbf{K}(k)$ denotes a complete elliptic integral of the first kind. Unfortunately, although well-adapted to numerical work, this expression is not particularly illuminating from an analytic perspective and we do not pursue it any further.

To summarize: we see that in $d=1$ space dimensions a lot can be said about the structure of the effective potential, without ever having to specify the precise nature of the driving term. We trust that the general outline of the calculation is clear, and that applying the method to specific examples will now be straightforward.

2.3. Reaction-Diffusion-Decay: $d=2$

In two spatial dimensions, reaction-diffusion equations have been extensively employed as models for the formation of patterns on animal coats (such as leopard spots), wing-marking patterns on butterflies, or fingerprint development in humans, etc.[1]

For $d=2$ the integral needed for the calculation of the effective potential is analytically tractable. After a change of variable, $x=vk^2$, and the introduction of an ultraviolet cutoff, Λ, the desired integral becomes

$$\int_0^{v\Lambda^2}\mathrm{d}x\,\sqrt{(x-P')^2+PP''}$$
$$=\tfrac{1}{2}P'\sqrt{(P')^2+PP''}-\tfrac{1}{2}PP''\ln[\sqrt{(P')^2+PP''}-P']$$
$$+\tfrac{1}{2}(v\Lambda^2-P')\sqrt{(v\Lambda^2-P')^2+PP''}$$
$$+\tfrac{1}{2}PP''\ln[\sqrt{(v\Lambda^2-P')^2+PP''}+v\Lambda^2-P'] \tag{46}$$

This integral is tabulated in ref. 27, Eqs. (2.261) and (2.262.1). For the time being we assume that all arguments of both the square roots and the logarithms are positive, and leave the more technical details for later. The integral that appears in the one-loop effective potential, Eq. (23), yields

$$\begin{aligned}
&\int_0^{\nu\Lambda^2} dx[\sqrt{(x-P')^2+PP''}-\sqrt{(x-P')^2}] \\
&\quad=\frac{1}{2}P'\sqrt{(P')^2+PP''}-\frac{1}{2}(P')^2-\frac{1}{2}PP''\ln[\sqrt{(P')^2+PP''}-P'] \\
&\quad+\frac{1}{2}(\nu\Lambda^2-P')^2\left[\sqrt{1+\frac{PP''}{(\nu\Lambda^2-P')^2}}-1\right] \\
&\quad+\frac{1}{2}PP''\ln(2\nu\Lambda^2)+\frac{1}{2}PP''\ln\left[\frac{\sqrt{(\nu\Lambda^2-P')^2+PP''}+\nu\Lambda^2-P'}{2\nu\Lambda^2}\right]
\end{aligned} \tag{47}$$

Taking the $\Lambda\to\infty$ limit, we get

$$\begin{aligned}
&\int_0^{\nu\Lambda^2} dx[\sqrt{(x-P')^2+PP''}-\sqrt{(x-P')^2}] \\
&\quad=\tfrac{1}{2}P'[\sqrt{(P')^2+PP''}-P'] \\
&\quad-\tfrac{1}{2}PP''\ln[\sqrt{(P')^2+PP''}-P'] \\
&\quad+\tfrac{1}{2}PP''\ln(2\nu\Lambda^2)+\tfrac{1}{4}PP''+O[1/(\nu\Lambda^2)]
\end{aligned} \tag{48}$$

This explicitly verifies the presence of the logarithmic term expected from naive power counting.[28] This logarithm is the only divergent contribution, and since it is proportional to $P(\phi)\,P''(\phi)$, the (one-loop) regularization may be performed by introducing the renormalization scale μ and making the following split into renormalized parameters and counterterms

$$P_{\text{bare}}(\phi)=P_{\text{renormalized}}(\phi)+\mathscr{A}KP''_{\text{renormalized}}(\phi)\ln(\Lambda^2/\mu^2) \tag{49}$$

where K is a calculable numerical constant whose precise value is not important for the present discussion. It is useful to make an additional finite renormalization in order to eliminate the $\frac{1}{4}PP''$ term from Eq. (48). (It is important to realise that these finite renormalizations do not affect the ground state structure. They are simply equivalent to a convenient choice of renormalization scale μ.)

After carrying out these steps, the one-loop effective potential becomes

$$\mathscr{V}[\phi;\phi_0;d=2]=\frac{1}{2}P^2(\phi)+\frac{\mathscr{A}}{16\pi v}\left[P'\left(\sqrt{(P')^2+PP''}-P'\right)\right.$$
$$\left.-PP''\ln\left(\frac{\sqrt{(P')^2+PP''}-P'}{v\mu^2}\right)\right]-(\phi\to\phi_0)+O(\mathscr{A}^2) \tag{50}$$

Like the action, the one-loop effective potential cannot explicitly depend on the renormalization scale μ. In fact, the renormalization group equation tells us that

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\mathscr{V}[\phi;\phi_0;d=2]=0\Rightarrow\mu\frac{\mathrm{d}P(\phi)}{\mathrm{d}\mu}=-\frac{\mathscr{A}}{8\pi v}P''(\phi)+O(\mathscr{A}^2) \tag{51}$$

which when combined with wavefunction renormalization and the general theorem of algebra will give the renormalization group equations for the couplings in $\mathscr{V}[\phi;\phi_0;d=2]$.[29] We note that this equation is similar in form to the one found for the $P(\phi)_2$ QFT in two spacetime dimensions, see ref. 29.

It is clear at this stage that the inclusion of a bare tadpole term $(P_0)_{\text{bare}}$ is essential. If there was not a tadpole, then $P(\phi)$ would start off as $P_1\phi+P_2\phi^2+P_3\phi^3+\cdots$, so that the lowest order term in the zero-loop potential would be $P_1^2\phi^2$. On the other hand, as is explicitly seen in Eq. (48), the divergent terms are proportional to $PP''=(P_1\phi+P_2\phi^2+P_3\phi^3+\cdots)(2P_2+6P_3\phi+\cdots)\to 2P_1P_2\phi+\cdots$, and there is a divergent term proportional to ϕ which is not present in the tree-level effective potential. Thus, in order to render the theory one-loop renormalizable a tadpole term must be included in the tree-level potential.

Notice that if $P(\phi)$ is odd $[P(-\phi)=-P(\phi)]$, then reaction-diffusion-decay is symmetric under the following Z_2 symmetry transformation

$$\phi\to-\phi\qquad\text{and}\qquad\eta\to-\eta \tag{52}$$

This prevents the generation of any of the even power monomial contributions to the polynomial $P(\phi)$, including the tadpole, and one never needs to introduce the tadpole at the tree-level. Notice however that this symmetry is not relevant to a realistic reaction-diffusion-decay system, since it excludes any two-body reactions (in fact all $2n$-body reactions).

Finally, we mention what happens when one has to deal with one of the branch cuts that we have temporarily suppressed for simplicity of

presentation. Although the intermediate stages of the calculation are algebraically involved (and messy), the ultimate answer is simple: take the real part of the expressions above.

2.4. Reaction-Diffusion-Decay: $d=3$

An example in which prediction and observation of pattern formation in three dimensions has created much interest is in the field of nonlinear optical systems. Recently, three-dimensional reaction-diffusion equations (of the Swift–Hohenberg type) have been derived for degenerate optical parametric oscillators in which three-dimensional Turing structures and spatial solitons have been predicted to exist as stable structures.[30] In addition, morphogenesis and structural development in embryos are examples of intrinsically three-dimensional phenomena.

The case $d=3$ is a straightforward generalization of the $d=1$ result. We are interested in evaluating

$$\begin{aligned}\mathscr{V}[\phi;\phi_0;d=3] = &\frac{1}{2}P^2(\phi)+\frac{1}{(2\pi)^2}\mathscr{A}\int_0^\infty \mathrm{d}k\, 4\pi k^2 \\ &\times\{\mathrm{Re}\sqrt{[\nu k^2-P'(\phi)]^2+P(\phi)\,P''(\phi)} \\ &-\sqrt{[\nu k^2-P'(\phi)]^2}\}-(\phi\to\phi_0)+O(\mathscr{A}^2)\end{aligned} \tag{53}$$

The integral is no longer finite and a single renormalization (without running logarithms) must be performed to absorb the infinity in the renormalized parameters. One must be careful to keep track of all the numerical coefficients and the final result is

$$\begin{aligned}\mathscr{V}[\phi;\phi_0;d=3] = &\frac{1}{2}P^2(\phi)+\frac{1}{(2\pi)^2}\mathscr{A}\frac{[-P'(\phi)]^{5/2}}{\nu^{3/2}} \\ &\times\sum_{n=1}^{\infty}\binom{1/2}{n}\frac{\Gamma(3/2)\,\Gamma(2n-5/2)}{2\Gamma(2n-1)} \\ &\times\left(\frac{P(\phi)\,P''(\phi)}{[P'(\phi)]^2}\right)^n-(\phi\to\phi_0)+O(\mathscr{A}^2)\end{aligned} \tag{54}$$

This expansion again only makes sense for $P'(\phi)<0$ (the region of field space in which small perturbations die away in the absence of noise). Furthermore, the radius of convergence of the resulting sum is one, so that this expression is limited to the region $PP''<(P')^2$. These limitations are

again not fundamental, but are merely a reflection of our choice of Taylor expansion in terms of the field variables. In a manner similar to what was done in $d=1$ we can also write the above Eq. (54) as

$$\mathscr{V}[\phi;\phi_0;d=3]=\frac{1}{2}P^2(\phi)+\frac{1}{(2\pi)^2}\mathscr{A}\frac{|P'(\phi)|^{5/2}}{\nu^{3/2}}F_{\pm}^{d=3}\left[\frac{P(\phi)\,P''(\phi)}{[P'(\phi)]^2}\right]-(\phi\to\phi_0)+O(\mathscr{A}^2) \tag{55}$$

with $F_{\pm}^{d=3}[z]$ now being the function

$$F_{\pm}^{d=3}[z]=\int_0^\infty \mathrm{d}x\,x^2\left[\mathrm{Re}\,\sqrt{(x^2\pm1)^2+z}-|x^2\pm1|-\frac{1}{2}\frac{z}{x^2}\right] \tag{56}$$

again such that $F_{\pm}[z=0]=0$. It is $F_+[z]$ that corresponds to the case $P'(\phi)<0$ discussed above [see Eq. (54)]. The case $F_-[z]$ is again trickier as there is no simple Taylor series expansion around $z=0$. Note that the last term z/x^2 is the counterterm introduced to guarantee UV finiteness, and that the integral is also IR finite. These functions can be analyzed in a manner analogous to the discussion for $d=1$ but for the sake of brevity we do not repeat details which are left to the industrious reader.

2.5. Special Cases: Monomial Interactions

There are nice simplifications for monomial interactions, where $P(\phi)=\xi\phi^n$. (Because of the generic presence of the tadpole term this monomial behaviour should always be imposed on the renormalized interactions, not the bare ones.) For monomial interactions the combination $P(\phi)\,P''(\phi)/[P'(\phi)]^2$ reduces to the constant $(n-1)/n$, and the one-loop effective potential (for $d=1$, and suppressing ϕ_0 for convenience) becomes

$$\mathscr{V}[\phi;d=1]=\frac{1}{2}\xi^2\phi^{2n}+\frac{1}{2\pi}\mathscr{A}\frac{|-\xi\phi^{n-1}|^{3/2}}{\nu^{1/2}}K_{\pm}[1;n]+O(\mathscr{A}^2) \tag{57}$$

with $K_{\pm}[1;n]$ a calculable (n-dependent) dimensionless constant. If the differential Eq. (6) (without noise) is assumed to be stable against small perturbations, the coupling ξ must be negative, and the exponent n must be an odd integer $n=2m+1$. In this case $P'(\phi)=(2m+1)\,\xi\phi^{2m}<0\Leftrightarrow\xi<0$ guarantees that linear time-dependent perturbations will decay in time. The effective potential becomes

$$\mathscr{V}[\phi;d=1]=\frac{1}{2}\xi^2\phi^{4m+2}+\frac{1}{2\pi}\mathscr{A}\frac{|\xi|^{3/2}\,|\phi|^{3m}}{\nu^{1/2}}K_{+}[1;2m+1]+O(\mathscr{A}^2) \tag{58}$$

From the previous equation, one can see that noise induced corrections to the effective potential dominate for small fields in one space dimension.

For two space dimensions the restriction to monomial interactions implies

$$\mathscr{V}[\phi; d=2] = \frac{1}{2}\xi^2\phi^{2n} + \frac{\mathscr{A}}{16\pi v}\xi^2\phi^{2n-2} \times \left\{ n\left(\sqrt{2n^2-n}-n\right) - n(n-1)\ln\left[\frac{\xi\phi^{n-1}(\sqrt{2n^2-n}-n)}{v\mu^2}\right]\right\} + O(\mathscr{A}^2) \tag{59}$$

By making a finite renormalization $\mu \to \bar{\mu}$, we can simplify the previous equation to obtain

$$\mathscr{V}[\phi; \phi_0; d=2] = \frac{1}{2}\xi^2\phi^{2n} - \frac{\mathscr{A}}{16\pi v}\xi^2\phi^{2n-2}n(n-1)\ln\left(\frac{\xi\phi^{n-1}}{v\mu^2}\right) + O(\mathscr{A}^2) \tag{60}$$

The coefficient of the logarithm is negative, implying a breakdown of perturbation theory for large fields. More importantly, since the coefficients of $P(\phi)$ run at one-loop, and a monomial is not a fixed point of the renormalization group equations, if we tune the interaction to be monomial at some fixed scale $\bar{\mu}$ then (in $d=2$) the interaction will not remain monomial if the scale is changed.

For three space dimensions the situation is similar to $d=1$. For a monomial interaction the one-loop effective potential is given by

$$\mathscr{V}[\phi; \phi_0; d=3] = \frac{1}{2}\xi^2\phi^{2n} + \frac{1}{(2\pi)^2}\mathscr{A}\frac{|-\xi\phi^{n-1}|^{5/2}}{v^{3/2}}K_{\pm}[3; n] + O(\mathscr{A}^2)$$

For a system stable in the absence of noise ($\xi < 0$ and $n = 2m+1$), we have

$$\mathscr{V}[\phi; \phi_0; d=3] = \frac{1}{2}\xi^2\phi^{4m+2} + \frac{1}{(2\pi)^2}\mathscr{A}\frac{|\xi|^{5/2}\,|\phi|^{5m}}{v^{3/2}}K_{+}[3; 2m+1] + O(\mathscr{A}^2)$$

In this case the noise induced effects (one-loop effects) become important for strong fields.

2.6. Special Cases: Higher Dimensions

We have seen that arbitrary polynomials in the reaction-diffusion-decay system subject to white noise are one-loop ultraviolet renormalizable

in 0, 1, 2, and 3 space dimensions (in fact, finite for 0 and 1 space dimensions). We can extend the range of dimensionalities in which these systems are one-loop renormalizable at the cost of restricting the form of the interaction.

For instance, in four space dimensions we have already seen that there is a divergence proportional to $P(\phi)\,P'(\phi)\,P''(\phi)$. For general $P(\phi)$ this cannot be renormalized, but if $P(\phi)$ is a polynomial of degree three or less, then $P'(\phi)\,P''(\phi)$ is also a polynomial of degree three or less. For this restricted class of interactions, the divergence can be absorbed into bare potential even in four space dimensions.

In six space dimensions there are two new divergent terms. They are proportional to $P(\phi)[P'(\phi)]^2\,P''(\phi)$ and $[P(\phi)\,P''(\phi)]^2$, respectively. If $P(\phi)$ is a polynomial of degree two or less then $[P'(\phi)]^2\,P''(\phi)$ is also a polynomial of degree two or less and the renormalization program can be carried out. In this case $P''(\phi)$ is either a constant or zero, so that the second type of divergence is no further obstruction.

Finally, in eight space dimensions there is only one new divergent term. It is proportional to $P(\phi)[P'(\phi)]^3\,P''(\phi)$ and so the theory is one-loop renormalizable only for linear interactions (i.e., for free fields where the theory is not only renormalizable but is actually finite.)

2.7. More General Noise

It is clear from the above arguments that the ultraviolet renormalizability of the reaction-diffusion-decay system depends critically on the large momentum behaviour of the noise two-point function. In some problems, colored noises maybe of interest, and we just give a necessarily brief discussion of the modifications needed in our analysis. Let us suppose that the noise is more general than (space-time) white noise. For instance, let us assume the noise is still temporally white, but spatially power-law distributed in the ultraviolet region with

$$\tilde{g}_2(\vec{k}) = \tilde{g}_2(|\vec{k}|) \approx (k/k_0)^{-\theta}\,\Theta(\Lambda - k) \tag{61}$$

where the positive exponent θ characterizes the strength of the ultraviolet-singular noise. In order to obtain the divergence structure of the one-loop effective potential, we must make use of Eq. (23). It is then easy to see that the first two terms in the expansion for the effective potential have ultraviolet behaviour proportional to $P(\phi)\,P''(\phi)\,\Lambda^{d-2-\theta}$, and $P(\phi)\,P'(\phi)\,P''(\phi)\,\Lambda^{d-4-\theta}$, respectively. Since the bare potential is $P^2(\phi)$, to have any hope of absorbing the infinities into the bare action we must have $d < 4 + \theta$. The one-loop

effective potential for the reaction-diffusion-decay system is then one-loop ultraviolet renormalizable for $d < 4 + \theta$ spatial dimensions.

If the noise is not temporally white, (but still Gaussian), one must revert to Eq. (18) and perform a case-by-case study.

3. NOISE AND STABILITY: AN APPLICATION

In this section we illustrate in broad strokes how the one-loop effective potential can be used to investigate the onset of instabilities and pattern formation in physical, biological, and chemical systems[1–3] modelled by reaction-diffusion equations (6).

By way of a concrete example which will serve as a simple template for the one-loop equation, we consider the following model:

$$\frac{\partial \phi}{\partial t} - v\vec{\nabla}^2\phi = P(\phi) + \eta(\vec{x}, t) \tag{62}$$

where $P(\phi) = a\phi^2 + b\phi + c$ is the kinetic reaction polynomial parameterized by three real constants a, b, c, and the diffusion constant v is a positive real number. We take the noise to be Gaussian and white. We first briefly run through the standard steps needed to perform a linear stability analysis of the noiseless or zero-loop version[10] of (62). This will serve as a point of reference when we come to discuss the linear stability analysis to be performed on the effective one-loop version of (62).

3.1. Zero-Noise Analysis

As is well known, the study of the onset of symmetry breaking instabilities, be they Hopf bifurcations or Turing instabilities, starts by classifying all the static and spatially homogeneous solutions ϕ_0 of the-reaction-diffusion equation at hand. These constant field configurations represent the maximally symmetric states of the system, which could be stable or unstable with respect to time and/or space dependent disturbances. For our toy model, these states satisfy

$$P(\phi_0) = 0 \tag{63}$$

[10] The noiseless limit of general reaction-diffusion equations need not necessarily coincide with the zero-loop limit, though these limits are in fact identical for our model equation; see the further comments to this effect below. For general details regarding the distinction between no-noise and zero-loops, see ref. 7.

which is the result of evaluating (62) for constant fields in the absence of noise (which is completely equivalent to the zero-loop limit of this equation). The solution is immediate, namely we have that

$$\phi_0^{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{64}$$

It should be pointed out that if the field ϕ represents a chemical concentration, then ϕ and ϕ_0 must be non-negative. There are many choices of the control parameters a, b, c for which this condition is met. With knowledge of the constant states, the next step is to expand about them so that we can study both the temporal and spatial evolution of disturbances with respect to these symmetric states. Setting $\varphi^{\pm} = \phi - \phi_0^{\pm}$, leads to

$$\frac{\partial \varphi^{\pm}}{\partial t} - v\vec{\nabla}^2 \varphi^{\pm} = \pm \varphi^{\pm} \sqrt{b^2 - 4ac} + a(\varphi^{\pm})^2 \tag{65}$$

which is the *exact* zero-noise equation for the perturbations. In arriving at this expression, we have used (63) and $P'(\phi_0^{\pm}) = \pm\sqrt{b^2 - 4ac}$, $P''(\phi_0^{\pm}) = 2a$. To study the onset of linear instabilities, we need only focus on the linear part of this equation. This is most conveniently carried out in momentum or mode space, for which we introduce the Fourier transform (in any number d of spatial dimensions) of the fluctuation $\varphi(\vec{x}, t)$

$$\varphi(\vec{x}, t) = \int \frac{\mathrm{d}^d \vec{q}}{(2\pi)^d} e^{i\vec{q}\cdot\vec{x}} \tilde{\varphi}_{\vec{q}}(t) \tag{66}$$

In terms of the mode functions $\tilde{\varphi}_{\vec{q}}(t)$, the (exact) zero-noise equation for the perturbations takes the form

$$\begin{aligned} \frac{\partial \tilde{\varphi}_{\vec{q}}}{\partial t} &= [\, -vq^2 \pm \sqrt{b^2 - 4ac}\,]\, \tilde{\varphi}_{\vec{q}} + a \int \frac{\mathrm{d}^d \vec{k}}{(2\pi)^d}\, \tilde{\varphi}_{\vec{k}} \tilde{\varphi}_{\vec{q}-\vec{k}} \\ &= \lambda(q^2)\, \tilde{\varphi}_{\vec{q}} + \text{nonlinearities} \end{aligned} \tag{67}$$

which identifies the momentum-dependent eigenvalue $\lambda(q^2)$ and the nonlinear mode-mode coupling terms.

3.1.1. Onset of Hopf Bifurcations. By definition, Hopf bifurcations are spatially homogeneous but time dependent linear instabilities.[1] Spatial homogeneity corresponds to the zero-momentum mode ($\vec{q} = \vec{0}$), so the onset of this class of instability is revealed by studying the properties

of the zero-momentum eigenvalue: $\lambda(q^2=0)$. A Hopf instability occurs when the real part $\mathrm{Re}\,\lambda(0)=0$. It is useful to consider the mode eigenvalues as functions of the control parameters (a, b, c) in the reaction kinetics polynomial. If we define b_{Hopf} by $b_{\mathrm{Hopf}}^2=4ac$, then it is easy to see that for $b<b_{\mathrm{Hopf}}$, the eigenvalue $\lambda(0)$ is pure imaginary and the linearized perturbations oscillate about the constant and static field configurations ϕ_0^+ and ϕ_0^- with the same frequency $\omega=|b^2-4ac|^{1/2}$. On the other hand, if $b>b_{\mathrm{Hopf}}$, then the linearized perturbations about ϕ_0^- decay exponentially, and they grow exponentially about ϕ_0^+. The instability therefore sets in at $b=b_{\mathrm{Hopf}}$, and only the ϕ_0^+ field configuration is unstable.

3.1.2. Onset of Turing Instabilities. The onset of spatial structure formation (Turing instabilities) occurs whenever the eigenvalue satisfies the condition $\mathrm{Re}[\lambda(q^2)]>0$, for some non-zero mode or modes.[1] The corresponding mode (or modes) set the length scale L (or scales) that characterize the spatial structures: $L\sim 1/q$. Note that for $b<b_{\mathrm{Hopf}}$, the real part of the eigenvalue for all modes is negative, so no spatial structure can form, and this holds for both the initial configurations $\phi_0^{\pm}$. Furthermore for ϕ_0^-, $b>b_{\mathrm{Hopf}}$, and for any $\vec{q}$, the eigenvalue $\lambda(q^2)$ is a negative real number. Therefore, no spatial patterns can develop as linear perturbations of this field configuration.

On the other hand, consider the case ϕ_0^+ and $b>b_{\mathrm{Hopf}}$. The eigenvalues for all modes are real, and of these, there is a finite band of momentum scales for which the eigenvalue is strictly positive: namely, for $0\leqslant q^2<q_T^2\equiv\sqrt{b^2-4ac}/v$. So, one expects onset of spatial structures to form with length scales corresponding to this momentum band.

This completes the purely linear stability analysis of the toy reaction-diffusion equation (62) *in the absence of noise*.

3.2. One-Loop Analysis

We now demonstrate, (by making use of the effective potential calculated in previous sections), how the inclusion of noise at one-loop impacts on the linear stability analysis illustrated above for our simple reaction-diffusion model. The idea is to repeat the above steps, but working now with the effective one-loop reaction-diffusion equation. To obtain the equations of motion in the presence of noise, we refer to the discussion in Appendix B of ref. 7. The complete effective action for reaction-diffusion systems in the presence of (arbitrary) Gaussian noise is given by

$$\Gamma[\phi;\phi_0]=\tfrac{1}{2}\iint \mathrm{d}^d\vec{x}\,\mathrm{d}t\,\mathrm{d}^d\vec{y}\,\mathrm{d}t'(D_{\mathrm{eff}}\phi-P_{\mathrm{eff}}[\phi])\,g_2^{-1}(D_{\mathrm{eff}}\phi-P_{\mathrm{eff}}[\phi]) \qquad (68)$$

where D_{eff} and P_{eff} are an effective differential operator, and effective kinetic reaction function, (not necessarily a polynomial!), respectively. The full (to all loops) equation of motion follows from the condition:[7]

$$\frac{\delta\Gamma[\phi;\phi_0]}{\delta\phi}=0 \qquad \text{or} \qquad \frac{\delta\mathscr{V}[\phi;\phi_0]}{\delta\phi}=0 \tag{69}$$

where the second expression holds for the effective potential and yields the dynamical equation satisfied by static and homogeneous field configurations. As a quick check of the formalism, consider the zero-loop effective potential: $\mathscr{V}[\phi;\phi_0]=\frac{1}{2}P^2[\phi]$. From (69) we must have $P(\phi)\,P'(\phi)=0$, so that *either* $P(\phi)=0$ and/or $P'(\phi)=0$. In fact, we know from (63) that it must be $P(\phi)$ that vanishes to correctly yield the solutions of (62) in the noise-free static and homogeneous field limit. The "spurious" factor $P'(\phi)$ is a consequence of the quadratic nature of the effective action (68). It is easy to verify that $P'(\phi)\neq 0$ evaluated at the zeroes of $P(\phi)$. This is important in that we do not generate more solutions than those corresponding to the zero-noise equation (62): zero-loops should correspond to zero-noise, and we find that this does in fact hold for our model. In general, spurious solutions will be absent whenever a certain differential operator $(D-P'[\phi])$ is non-singular (invertible). Even in those situations where it can become singular, a simple limiting procedure can be invoked to eliminate the spurious solutions.[7]

Under these conditions, i.e., making use of the invertibility of $(D-P'[\phi])$, the one-loop equation of motion associated with (62) is

$$\frac{\partial\phi}{\partial t}-v\vec{\nabla}^2\phi+O(\mathscr{A}\vec{\nabla}^2\phi,\,\mathscr{A}\partial_t^2\phi,\ldots)=(2\mathscr{V}[\phi;\phi_0])^{1/2}$$

$$=[P^2(\phi)+2\mathscr{A}X(\phi)+O(\mathscr{A}^2)]^{1/2} \tag{70}$$

where the one-loop terms in the effective potential denoted above by $X(\phi)$ have been computed for general kinetic reaction functions and may be written down by inspection from (34), (40), (50), and (54) above. It should be noted that the one-loop contribution to the effective potential vanishes whenever $P(\phi)=0$ so that without loss of generality we can write the $O(\mathscr{A})$ terms as

$$X(\phi)=P(\phi)\,h(\phi) \tag{71}$$

with $h(\phi)$ remaining finite (and generally non-zero) as one approaches solutions of the zero-noise equations of motion. Furthermore, at one-loop

we know from (rather general considerations detailed in ref. 28) that the one-loop effective action for any reaction-diffusion equation driven by white noise has no wavefunction renormalization in fewer than six spatial dimensions. This means that in dimension six or less, the differential operator *D per se* remains unchanged at one-loop. There can, however, be *finite* renormalizations that induce finite and calculable one-loop corrections that involve combinations of fields and derivatives; we summarize the first few possible structures belonging to this class above. It is clear from (70) that as the noise amplitude $\mathscr{A}$ is taken to zero, we recover the zero-loop (noiseless) equation (62) of motion.

In view of the above, the one-loop equations of motion read

$$\frac{\partial\phi}{\partial t} - \nu\vec{\nabla}^2\phi + O(\mathscr{A}\vec{\nabla}^2\phi, \mathscr{A}\partial_t^2\phi, \ldots) = P(\phi)\left[1 + \mathscr{A}\frac{2h(\phi)}{P(\phi)} + O(\mathscr{A}^2)\right]^{1/2}$$
$$= P(\phi) + \mathscr{A}h(\phi) + O(\mathscr{A}^2) \tag{72}$$

We are now ready to proceed with the stability analysis for the one-loop reaction-diffusion equation (70). Just as for the noiseless, zero-loop case, one begins by solving for all the possible static and spatially homogeneous configurations. These will be the solutions of

$$P(\hat{\phi}_0) + \mathscr{A}h(\hat{\phi}_0) + O(\mathscr{A}^2) = 0 \tag{73}$$

no matter how complicated the derivative structure on the left-hand side of the one-loop equations of motion may be. We denote by $\hat{\phi}_0$ the one-loop constant field configurations to distinguish them from their tree-level counterparts. In fact, it is easy to demonstrate that the zero-loop solutions ϕ_0 are in general *not* solutions of this one-loop equation (73). Thus, we can conclude that typically $\hat{\phi}_0 \neq \phi_0$. Physically, this reflects the fact that the presence of noise has altered the symmetric states of the system. Suppose we have chosen a particular spatial dimension and have catalogued these new symmetric states. [To do so in actual practice requires selecting one of (34), (40), (50), and (54), and solving the resulting (algebraic-transcendental) equation implied by (73). However, bear in mind that the point we wish to make can be achieved without doing so explicitly.] The next step involves expanding the one-loop equation in (linear) perturbations about these one-loop states. We define $\hat{\varphi} = \phi - \hat{\phi}_0$, and write

$$\frac{\partial\hat{\varphi}}{\partial t} - \nu\vec{\nabla}^2\hat{\varphi} + O(\mathscr{A}\vec{\nabla}^2\hat{\varphi}, \mathscr{A}\partial_t^2\hat{\varphi}, \ldots) = P(\hat{\varphi} + \hat{\phi}_0) + \mathscr{A}h(\hat{\varphi} + \hat{\phi}_0) + O(\mathscr{A}^2) \tag{74}$$

We Taylor expand the right-hand side of (74) making use of (73). We then obtain the linearized one-loop equation of motion

$$\frac{\partial \hat{\varphi}}{\partial t} - \nu \vec{\nabla}^2 \hat{\varphi} + O(\mathscr{A}\vec{\nabla}^2 \hat{\varphi}, \mathscr{A}\partial_t^2 \hat{\varphi}, \ldots)$$
$$= [P'(\hat{\phi}_0) + \mathscr{A} h'(\hat{\phi}_0)]\, \hat{\varphi} + O(\mathscr{A}^2) + \text{nonlinearities} \tag{75}$$

After Fourier transformation the evolution of the mode functions is governed by

$$\frac{\partial \tilde{\hat{\varphi}}_{\vec{q}}}{\partial t} = \hat{\lambda}(q^2)\, \tilde{\hat{\varphi}}_{\vec{q}} + O(\mathscr{A}^2) + \text{nonlinearities-}O(\mathscr{A}\partial_t^2 \tilde{\hat{\varphi}}_{\vec{q}}, \ldots) \tag{76}$$

where the one-loop mode-dependent eigenvalues are

$$\hat{\lambda}(q^2) = -\nu q^2 + P'(\hat{\phi}_0) + \mathscr{A} h'(\hat{\phi}_0) - O(\mathscr{A} q^2, \ldots) \tag{77}$$

The function h is the one-loop contribution to the effective potential (divided by P) and has been explicitly calculated (for arbitrary P) in various spatial dimensions. Since it is a function of the reaction polynomial, it is also a function of the control parameters (a, b, c) appearing in P. If there are finite renormalizations leading to new derivative structures, then $\hat{\lambda}$ will also depend on their respective numerical coefficients. For the toy model considered here, this means that the one-loop mode eigenvalues $\hat{\lambda}$ are also functions of the same parameters (a, b, c) that appeared in the zero-loop eigenvalues in (67), as well as of whatever finite renormalizations are present at one-loop.

We summarize our main point: when we investigate the conditions for the onset of Hopf and Turing instabilities at one-loop, $\mathrm{Re}[\hat{\lambda}(0)] = 0$ and $\mathrm{Re}[\hat{\lambda}(q^2)] > 0$, $q^2 \neq 0$, we will obtain new conditions on the parameters that govern the onset of whatever instabilities are present, with respect to the $O(\mathscr{A})$ noise-altered symmetric states $\hat{\phi}_0$. We see that the effect of the noise is to shift the symmetric states of the system, as well as to change the nature of the linear instabilities that may be induced by perturbations around these new states.

4. DISCUSSION

In this paper we have made use of the *field theory* formalism developed in ref. 7 and have applied it to the class of stochastic partial differential equations subject to diffusion, decay, and polynomial reaction kinetics. These equations are, for example, extensively considered in the

mathematical modeling of chemical reactions, and biological pattern formation.[1–5]

We have adopted a purely phenomenological strategy and assumed the form of the stochastic PDE to be given by Eq. (11), driven by real additive noise with correlation function (2). We have then applied quantum field theory methods to this equation in order to obtain the corresponding (one-loop) effective potential. Although recognizing the possibility to map microphysics to an effective stochastic PDE[13–15] when the underlying microscopic master equation is *known*, this does not diminish the usefulness of starting from phenomenological equations to model complex systems, where either knowledge of the complete microphysics is lacking or where one can not account for all sources of noise, both internal and external, and fluctuations. As a case in point, we mention the problem of predicting the pattern formation of animal-coat markings starting from the molecular biology of genes: even if this formidable task could be carried out, there are a host of truly unpredictable time and space dependent contingencies that influence the animal's development in the course of pattern formation, and these certainly cannot be derived from first principles nor deterministically anticipated. But we can certainly model them with (real) noise, which could be additive and/or multiplicative. Once we have the candidate stochastic PDE in hand, we can go on to calculate its effective potential, a poweful tool permiting us to develop additional physical insight.

The subtleties of the physical interpretation and the usefulness of the effective potential for non-equilibrium dynamics have already been addressed elsewhere;[7] here we apply the effective potential formalism to a specific class of real reaction- diffusion equations.

We have restricted that framework to the case of white noise and have calculated the one-loop corrections (i.e., we have taken into account the second-order fluctuations about the static and homogeneous solutions of these equations) to the effective potential in various spatial dimensions and for a general polynomial reaction kinetics term $P(\phi)$. The effective potential, which provides information about the possible ground states of the system (which may or may not be stable to small perturbations, see below), is calculated by functional integral methods and we find that it is one-loop finite for zero and one space dimensions, and one-loop renormalizable in two and three space dimensions. By *finite* we mean there are no short-distance divergences, and by *renormalizable*, we mean that whatever ultraviolet divergences are present, they can be absorbed into the parameters appearing in the original stochastic partial differential equation. In particular, in two space dimensions, this renormalizability leads to a set of one-loop renormalization group equations (51) that govern the renormalization-scale dependence of the parameters present in the original reaction polynomial.

In understanding the onset of spatio-temporal pattern formation in systems out of equilibrium, it has proven extremely useful to begin the analysis by first solving for and then classifying all the static and spatially homogeneous states allowed by the time-dependent partial differential equations employed to model the system in question.[31] In this way, one can straightforwardly decide whether the system will exhibit Hopf bifurcations and/or Turing instabilities and get a handle on the qualitative nature of the pattern expected to emerge.

This can be followed up by an amplitude analysis of the *fluctuations* about these static and homogeneous states. The unstable modes are the ones that lead to non-trivial patterns. For out-of-equilibrium systems coupled to noisy environments (or with inherent internal noise) it is important to know how the stochastic sources can alter and shift these static and homogeneous states, since these affect the onset of the pattern-forming (linear) instabilities. It is seen that indeed noise will affect these patterns in a way which is computable. In fact, the calculations in this paper show how the effects of stochastic noise on these states of reaction-diffusion systems can be taken into account in an elegant and computationally direct way following the general formalism developed in ref. 7.

APPENDIX A. JACOBIAN FUNCTIONAL DETERMINANT IN ZERO SPACE DIMENSIONS

We are interested in evaluating the following functional determinant in zero space dimensions

$$\mathscr{J} \equiv \det\left[(\partial_t)^n - \frac{\mathrm{d}P}{\mathrm{d}\phi} \right] \tag{78}$$

If $n > 1$, from the general analysis given in ref. 7, we see

$$\mathscr{J}_n = \det(\partial_t)^n \tag{79}$$

whereas for $n = 1$, we get[7]

$$\mathscr{J}_1 = \det(\partial_t) \exp\left(-\Theta(0) \frac{\mathrm{d}P}{\mathrm{d}\phi} \right) \to \det(\partial_t) \exp\left(-\frac{1}{2} \frac{\mathrm{d}P}{\mathrm{d}\phi} \right) \tag{80}$$

where we have adopted the prescription $\Theta(0) = 1/2$.

APPENDIX B. JACOBIAN FUNCTIONAL DETERMINANT FOR THE REACTION-DIFFUSION-DECAY SYSTEM

For the reaction-diffusion-decay system $F[\phi(\vec{x}, t)] = P(\phi)$ is an ultra-local function of ϕ, most commonly a polynomial in ϕ. Thus

$$\frac{\delta F[\phi(\vec{x}, t)]}{\delta \phi(\vec{y}, t')} \to P'(\phi(\vec{x}))\, \delta(\vec{x} - \vec{y})\, \delta(t - t') \tag{81}$$

The relevant trace is given by[7]

$$\begin{aligned} \mathrm{Tr}\left[G_1 \frac{\delta F[\phi(\vec{x})]}{\delta \phi(\vec{y})} \right] &= \Theta(0) \int \mathrm{d}t\, \mathrm{tr}_{\mathrm{space}}[P'(\phi)\, \delta(\vec{x} - \vec{y})] \\ &= \Theta(0)\, \delta^d(\vec{0}) \int \mathrm{d}^d\vec{x}\, \mathrm{d}t\, P'(\phi) \end{aligned} \tag{82}$$

In contrast to the Kardar–Parisi–Zhang equation,[26] the Jacobian determinant for the reaction-diffusion-decay system is *not* a field independent constant, but (adopting the prescription $\Theta(0) = 1/2$) one has

$$\mathscr{J}_{RDD} = \exp\left[-\tfrac{1}{2}\, \delta^d(\vec{0}) \int \mathrm{d}^d\vec{x}\, \mathrm{d}t\, P'(\phi) \right] \tag{83}$$

Thus for the reaction-diffusion-decay field theory the functional determinant at worst leads to extremely simple Faddeev–Popov ghosts. There are general arguments (see for example Zinn–Justin,[25] pp. 373, 307, or related comments in Itzykson–Zuber,[24] p. 448) to the effect that terms proportional to $\delta^d(\vec{0})$ can always be safely discarded in dimensional regularization. We do not want to step into the middle of this contentious issue and merely note that we have found it more convenient to *not* adopt the formal result $\delta^d(\vec{0}) = 0$, and instead to explicitly carry the Jacobian along in the calculation. Keeping the Jacobian explicit is essential to showing one-loop finiteness in $d = 1$ space dimension, a result that would otherwise be disguised by unnecessarily discarding the precise counterterm needed to ensure the one-loop finiteness of the theory.

This situation is in marked contrast to that for the KPZ system,[26] wherein the Jacobian determinant is a field independent constant irrespective of how one wishes to handle the formal result $\delta^d(\vec{0}) = 0$.

APPENDIX C. FEYNMAN RULES FOR THE REACTION-DIFFUSION-DECAY SYSTEM

We have derived the general form of the Feynman rules in the direct formalism, applicable to arbitrary SPDEs, in ref. 7. There are a number of technical simplifications for the reaction-diffusion-decay system which make it worthwhile to present this particular case. From the reaction-diffusion-decay stochastic differential equation (6)

$$\left[\frac{\partial}{\partial t} - \nu(\vec{\nabla}^2 - m^2)\right] \phi = P(\phi) + \eta \tag{84}$$

we deduce the characteristic functional (partition function)[7]

$$Z[J] = \int (\mathscr{D}\phi) \sqrt{\mathscr{J}\mathscr{J}^\dagger} \exp\left(-\tfrac{1}{2} \iint [\partial_t \phi - \nu(\vec{\nabla}^2 - m^2)\,\phi - P(\phi)]\; G_\eta^{-1} \right.$$
$$\left. \times [\partial_t \phi - \nu(\vec{\nabla}^2 - m^2)\,\phi - P(\phi)] \right) \exp\left(\int [J\phi] \right) \tag{85}$$

from which we can immediately deduce the associated Feynman rules. Note we have opted to collect whatever terms linear in ϕ there may be in the reaction polynomial $P(\phi)$ and place them on the left hand side of the equation. Compare to the Eqs. (6) and (11) in the introduction: the decay rate is simply $\gamma = \nu m^2$. The functional determinant can either be calculated from the preceding section, or it can be included as a Faddeev–Popov ghost term, with the same effect. There is only one field propagator and two vertices. The propagator is identical to the KPZ case, but the vertices have as high an order as determined by the highest power in P (see below). The following holds for translationally invariant noise.

These are the basic structures (in the interacting classical action) from which one can derive the appropriate vertices by taking the corresponding functional derivatives:

Propagator:

$$G_{\text{field}}(\vec{k}, \omega) = \frac{G_\eta(\vec{k}, \omega)}{\omega^2 + \nu^2(\vec{k}^2 + m^2)^2}$$

$P(\phi) - \phi$ vertex:

$$\frac{[\tilde{P}(\phi)](\vec{k}_1, \omega_1)\, \tilde{\phi}(\vec{k}_2, \omega_2)}{(2\pi)^{d+1}} \frac{[-i\omega_1 + \nu(\vec{k}_1^2 + m^2)]}{G_\eta(\vec{k}_1, \omega_1)} \delta(\vec{k}_1 + \vec{k}_2)\, \delta(\omega_1 + \omega_2)$$

$P(\phi) - P(\phi)$ vertex:

$$\frac{1}{2} \frac{[\tilde{P}(\phi)](\vec{k}_1, \omega_1)[\tilde{P}(\phi)](\vec{k}_2, \omega_2)}{(2\pi)^{d+1} G_\eta(\vec{k}_1, \omega_1)} \delta(\vec{k}_1 + \vec{k}_2)\, \delta(\omega_1 + \omega_2)$$

For a polynomial $P(\phi)$ of order n, there will be up to $2n$ propagators meeting at a vertex. Calculations with these Feynman rules involve fewer propagators and vertices than those of the Martin–Siggia–Rose formalism.[8, 9] The basic trade-of is this: one reduces the number of propagators and vertices at the cost of making the vertices more complicated. As long as one is interested in one-loop physics this cost is not prohibitive,[7] and the pay-off in terms of ease of calculation for the effective action and effective potential is considerable.

APPENDIX D. AN INTEGRAL

We want to demonstrate that

$$\int_{-\infty}^{+\infty} d\omega \ln\left(\frac{\omega^2 - Z^2 \pm i\varepsilon}{\omega^2 + Y^2}\right) = 2\pi(\pm iZ - Y) \tag{86}$$

To see this note that

$$\begin{aligned}\int_{-\infty}^{+\infty} d\omega \ln\left(\frac{\omega^2 - Z^2 \pm i\varepsilon}{\omega^2 + Y^2}\right) &= \int_{-\infty}^{+\infty} d\omega \ln\left(\frac{|\omega^2 - Z^2|}{\omega^2 + Y^2}\right) \pm i\pi \int_{-Z}^{+Z} d\omega \\ &= \int_{-\infty}^{+\infty} d\omega \ln\left(\frac{|\omega^2 - Z^2|}{\omega^2 + Y^2}\right) \pm 2\pi i Z \end{aligned} \tag{87}$$

The remaining integral can be found in ref. 27, Eqs. (2.736.1) and (2.733.1), and we can finally write

$$\begin{aligned}&\int_{-\infty}^{+\infty} d\omega \ln\left(\frac{|\omega^2 - Z^2|}{\omega^2 + Y^2}\right) \\ &\quad = \left\{\omega \ln\left(\frac{|\omega^2 - Z^2|}{\omega^2 + Y^2}\right) + Z \ln\left|\frac{\omega + Z}{\omega - Z}\right| - 2Y \tan^{-1}\left(\frac{\omega}{Y}\right)\right\}\Bigg|_{-\infty}^{+\infty} \\ &\quad = -2\pi Y \end{aligned} \tag{88}$$

APPENDIX E. THE ITO CALCULUS VERSUS THE STRATONOVICH CALCULUS

In evaluating the Jacobian functional determinant one encounters a factor of $\Theta(0)$, which is ill-defined and must be specified by some prescription. The prescription which is most useful in this context, and which we have adopted in the bulk of this paper, is the symmetric prescription wherein $\Theta(0)$ equals $\frac{1}{2}$. This may be justified by a limiting procedure as described, for example, in the text by Zinn–Justin[25] (Chapter 4, pp. 69–70.) The symmetric prescription is equivalent to adopting the *Stratonovich calculus* for stochastic equations. Choosing $\Theta(0)=0$ is equivalent to the *Ito calculus*. The Ito calculus simplifies the Jacobian determinant (to unity) at the cost of destroying equivariance under field redefinitions (the Ito calculus explicitly breaks coordinate invariance in field space). See, for instance, Eyink,[11] or Zinn–Justin.[25] In this Appendix we sketch the modifications required to implement the Ito calculus. These changes are straightforward if at times tricky (the loss of reparameterization invariance under field redefinitions implies that all arguments involving a change of variables must be carefully re-assessed).

If we adopt the Ito calculus, then for any SPDE the (unrenormalized) expression for the one-loop effective potential simplifies to

$$\begin{aligned}\mathscr{V}_{\text{Ito}}[\phi;\phi_0] = {} & \frac{1}{2}F^2[\phi] + \frac{1}{2}\mathscr{A}\int \frac{\mathrm{d}^d\vec{k}\,\mathrm{d}\omega}{(2\pi)^{d+1}} \\ & \times \ln\left[\left(D^\dagger(\vec{k},\omega) - \frac{\delta F^\dagger}{\delta\phi}\right)\left(D(\vec{k},\omega) - \frac{\delta F}{\delta\phi}\right)\right. \\ & \left. + \tilde{g}_2(\vec{k},\omega)\,F[\phi]\,\frac{\delta^2 F}{\delta\phi\,\delta\phi}\right] - (\phi\to\phi_0) + O(\mathscr{A}^2) \end{aligned} \tag{89}$$

[Compare with Eq. (4).] When specialized to the reaction-diffusion-decay system this further simplifies to

$$\begin{aligned}\mathscr{V}_{\text{Ito}}[\phi;\phi_0] = {} & \frac{1}{2}\left[P^2(\phi) - P^2(\phi_0)\right] + \frac{1}{2}\mathscr{A}\int \frac{\mathrm{d}^d\vec{k}\,\mathrm{d}\omega}{(2\pi)^{d+1}} \\ & \times \ln\left[\frac{\omega^2 + [\nu\vec{k}^2 - P'(\phi)]^2 + \tilde{g}_2(\vec{k},\omega)\,P(\phi)\,P''(\phi)}{\omega^2 + [\nu\vec{k}^2 - P'(\phi_0)]^2 + \tilde{g}_2(\vec{k},\omega)\,P(\phi_0)\,P''(\phi_0)}\right] \\ & + O(\mathscr{A}^2) \end{aligned} \tag{90}$$

[Compare with Eq. (19).] Further restricted to temporally white noise we obtain

$$\mathscr{V}_{\text{Ito}}[\phi;\phi_0] = \frac{1}{2}P^2(\phi) + \frac{1}{2}\mathscr{A}\int\frac{d^d\vec{k}}{(2\pi)^d} \times\{\text{Re}[\sqrt{[v\vec{k}^2 - P'(\phi)]^2 + \tilde{g}_2(\vec{k})\,P(\phi)\,P''(\phi)}]\} - (\phi\to\phi_0) + O(\mathscr{A}^2) \tag{91}$$

[Compare with Eq. (23).]

On the one hand, this looks like a tremendous simplification of Eq. (23). On the other hand, the ultraviolet renormalizability properties are now considerably worse. For instance, by inspection of the above it is easy to see that one can no longer rely on a cutoff in the noise spectrum, $g_2(\vec{k})$, to keep the effective potential finite. Instead a cutoff in the momentum integral must be introduced by hand, complicating the process tremendously. Even if $g_2(\vec{k})\to 0$ for large momenta, for $d>0$ there exists an UV divergence proportional to $\Lambda^d[P'(\phi)-P'(\phi_0)]$. For generic $P(\phi)$ this cannot be absorbed into a counterterm in the zero-loop effective action. That is, adopting the Ito calculus for the reaction-diffusion-decay system results in a theory that is one-loop non-renormalizable for $d>0$, and so must be viewed as an "effective field theory."

Thus even for $d=1$, where the Stratonovich calculus leads to a one-loop finite result, the Ito calculus leads to complicated expressions which obscure the underlying physics. It is for this reason, (the complications of dealing with non-renormalizable effective field theories), and the fact that the Ito calculus is not invariant under field redefinitions, that we have not further explored the Ito calculus in this paper.

The one case where the Ito calculus gives a simpler answer than the Stratonovich calculus is for $d=0$, corresponding to stochastic mechanics rather than stochastic field theory. In that case

$$\mathscr{V}_{\text{Ito}}[\phi;\phi_0;d=0] = \tfrac{1}{2}P^2(\phi) + \tfrac{1}{2}\mathscr{A}(\text{Re}\sqrt{\tfrac{1}{2}[P^2(\phi)]''}) - (\phi\to\phi_0) + O(\mathscr{A}^2) \tag{92}$$

Of course, one should not be alarmed that the Ito calculus and the Stratonovich calculus give different intermediate results; they are different theories. Because the Ito calculus is not invariant under field redefinitions it is generally possible to find *some* choice of field variables (a reparametrization) that makes the two systems agree with each other, but that special set of field variables may not be the ones one naively started out with.

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